Imperial College London

Lecture 13

Sampling & Discrete signals
(Lathi 8.1-8.2)

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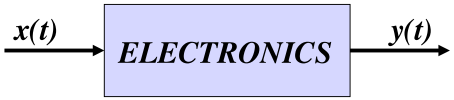
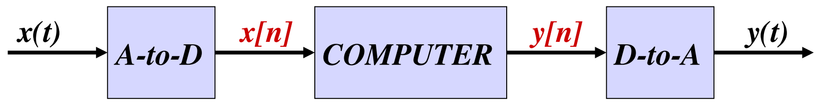
Continuous time vs Discrete time

- Continuous time system
  - Good for analogue & general understanding
  - Appropriate mostly to analogue electronic systems



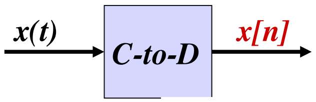
- Electronics are increasingly digital
  - E.g. mobile phones are all digital, TV broadcast is will be 100% digital in UK
  - We use digital ASIC chips, FPGAs and microprocessors to implement systems and to process signals
  - Signals are converted to numbers, processed, and converted back

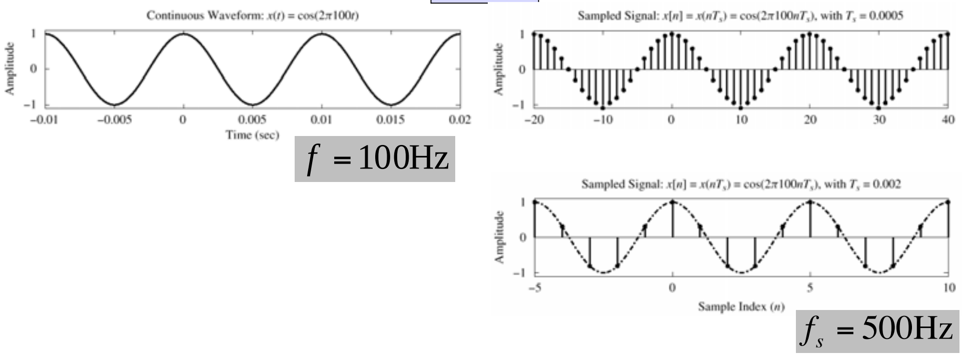


Sampling Process

- Use A-to-D converters to turn x(t) into numbers x[n]
- Take a sample every sampling period T_s – uniform sampling

  $x[n] = x(nT_s)$





Sampling Theorem

- Bridge between continuous-time and discrete-time
- Tell us HOW OFTEN WE MUST SAMPLE in order not to loose any information

Sampling Theorem

A continuous-time signal x(t) with frequencies no higher than f_{max} (Hz) can be reconstructed EXACTLY from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

- For example, the sinewave on previous slide is 100 Hz. We need to sample this at higher than 200 Hz (i.e. 200 samples per second) in order NOT to loose any data, i.e. to be able to reconstruct the 100 Hz sinewave exactly.
- fmax refers to the maximum frequency component in the signal that has **significant** energy.
- Consequence of violating sampling theorem is **corruption of the signal** in digital form.

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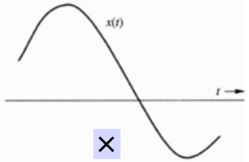
Sampling Theorem: Intuitive proof (1)

- Consider a bandlimited signal x(t) and is spectrum X(ω):



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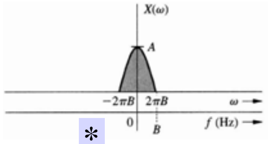
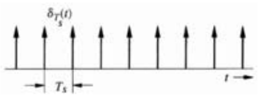


$\times$ $*$

- Ideal sampling = multiply x(t) with impulse train (Lec 10/12):



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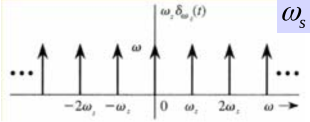
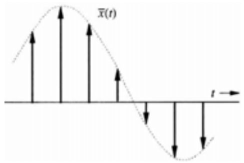


$\omega_s = 2\pi / T_s$

- Therefore the sampled signal has a spectrum:



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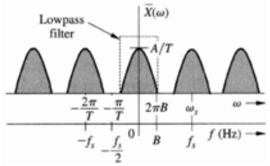

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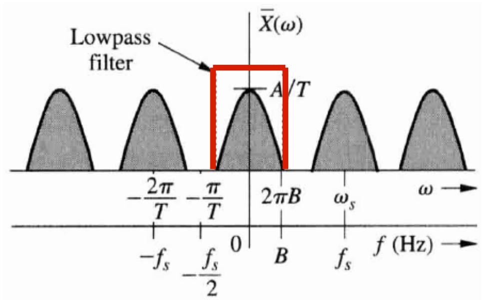
Sampling Theorem: Intuitive proof (2)

- Therefore, to reconstruct the original signal x(t), we can use an ideal lowpass filter on the sampled spectrum:



- This is only possible if the shaded parts do not overlap. This means that fs must be more than TWICE that of B.

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Sampling Theorem: mathematical proof

- The sampled version can be expressed as:

$$\bar{x}(t) = x(t)\delta_{Ts}(t) = \sum_n x(nT_s)\delta(t - nTs)$$

- We can express the impulse train as a Fourier series:

$$\delta_{Ts}(t) = \frac{1}{Ts}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + ...] \quad \text{where} \quad \omega_s = 2\pi / Ts$$

- Therefore:

$$\bar{x}(t) = \frac{1}{Ts}[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + ...]$$

- Since $\quad 2x(t)\cos\omega_s t \quad \Leftrightarrow \quad X(\omega - \omega_s) + X(\omega + \omega_s)$

$$\bar{X}(\omega) = \frac{1}{Ts}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

- Which is essentially the spectrum shown in the previous slide.

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Whose theorem is this?

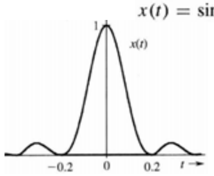
- The sampling theorem is usually known as the **Shannon Sampling Theorem** due to Claude E. Shannon's paper "A mathematical theory of communciation" in 1948. However, he himself said that "… is common knowledge in the communication art."
- The minimum required sampling rate fs (i.e. 2xB) is known as the **Nyquist sampling rate** or **Nyquist frequency** because of H. Nyquist's work on telegraph transmission in 1924 with K. Küpfmüller.
- The first formulation of the sampling theorem precisely and applied it to communication is probably a Russian scientist by the name of V. A. Kotelnikov in 1933.
- However, mathematician already knew about this in a different form and called this the interpolation formula. E. T. Whittaker published the paper "On the functions which are represented by the expansions of the interpolation theory" back in 1915!

L8.1 p771

What happens if we sample too slowly?

- What are the effects of sampling a signal at, above, and below the Nyquist rate? Consider a signal bandlimited to 5Hz:

$x(t) = \text{sinc}^2(5\pi t)$ $\qquad$ $X(\omega) = 0.2\,\Delta(\omega/20\pi)$



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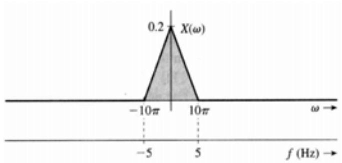
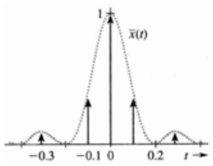


- Sampling at Nyquist rate of 10Hz give:

perfect reconstruction possible



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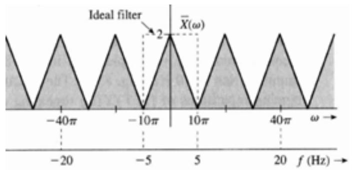

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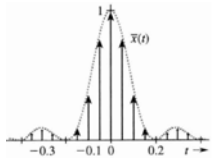
What happens if we sample too slowly?

- Sampling at higher than Nyquist rate at 20Hz makes reconstruction much easier.

perfect reconstruction practical



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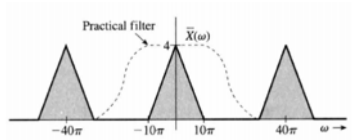
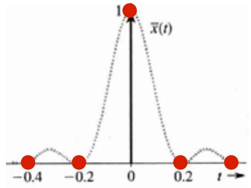


- Sampling below Nyquist rate at 5Hz corrupts the signal.



ALIASING

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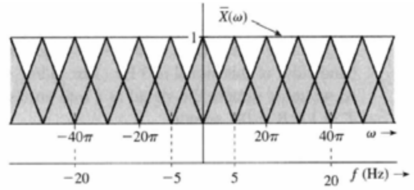

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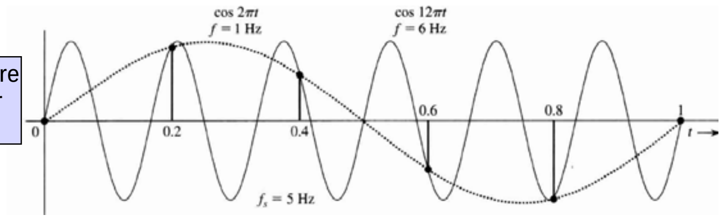
Spectral folding effect of Aliasing

- Consider what happens when a 1Hz and a 6Hz sinewave is sampled at a rate of 5Hz.

1Hz & 6Hz sinewaves are indistinguishable after sampling



- In general, if a sinusoid of frequency f Hz is sampled at fs samples/sec, then sampled version would appear as samples of a continuous-time sinusoid of frequency $|f_a|$ in the band 0 to fs/2, where:

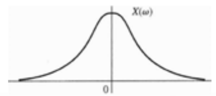
$$|f_a| = |f - mf_s| \qquad \text{where} \quad |f_a| \le \frac{f_s}{2}, \ \text{m is an integer}$$

- In other words, the 6Hz sinewave is FOLDED to 1Hz after being sampled at 5Hz.

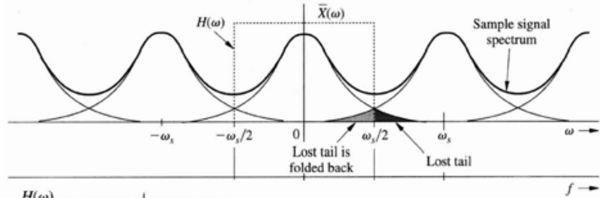
L8.2 p786

Anti-aliasing filter (1)

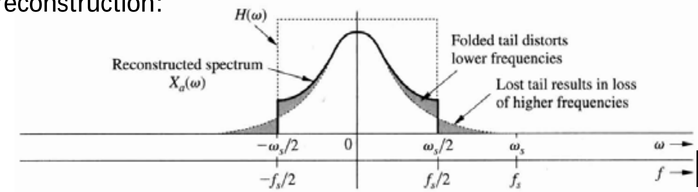
- To avoid corruption of signal after sampling, one must ensure that the signal being sampled at fs is bandlimited to a frequency B, where B < fs/2.
- Consider this signal spectrum:



- After sampling:



- After reconstruction:



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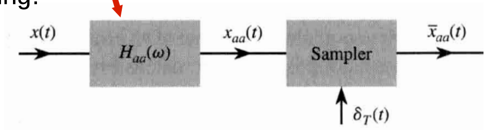
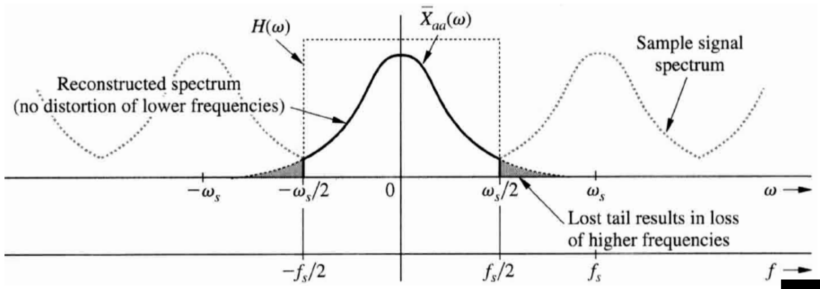
Anti-aliasing filter (2)

- Apply a lowpass filter before sampling:



- Now reconstruction can be done without distortion or corruption to lower frequencies:



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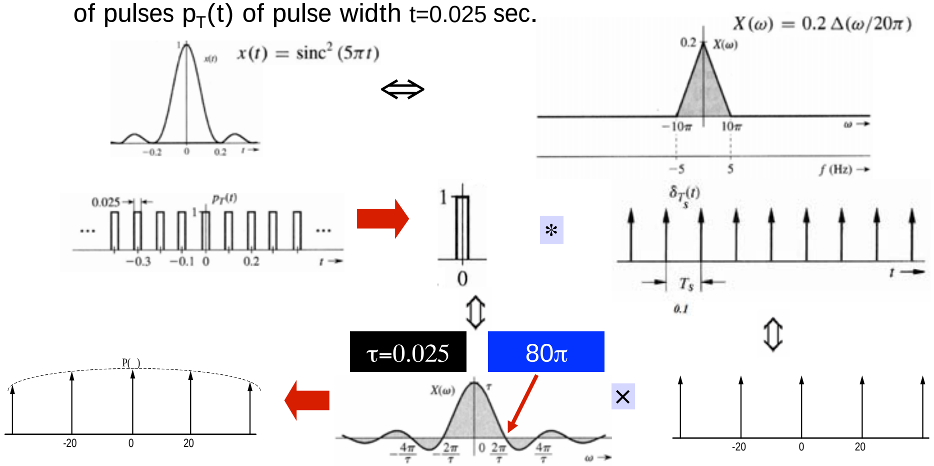
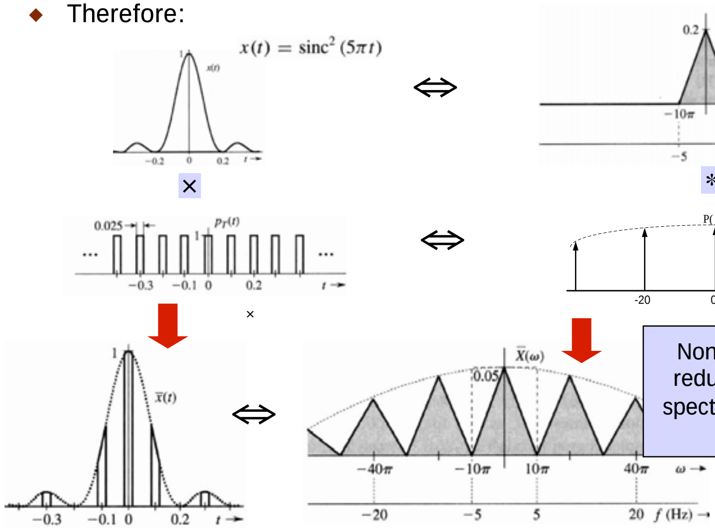
Practical Sampling (1)

- Impulse train is not a very practical sampling signal. Let us consider a train of pulses $p_T(t)$ of pulse width t=0.025 sec.



Practical Sampling

- Therefore:



Non-ideal sampling only reduces gain of repeated spectra. Sampling theorem still holds!

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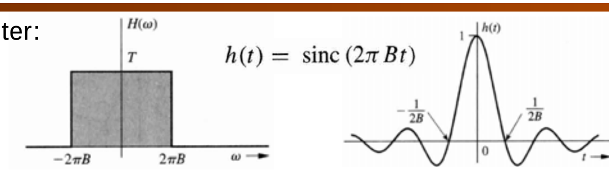
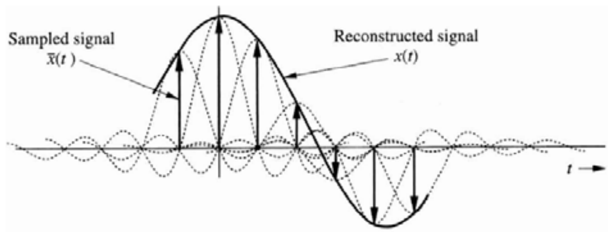
Ideal Signal Reconstruction

- Use ideal lowpass filter:



- That's why the sinc function is also known as the **interpolation** function:



$$x(t) = \sum_n x(nT)h(t - nT)$$

$$= \sum_n x(nT)\,\mathrm{sinc}\,(2\pi Bt - n\pi)$$

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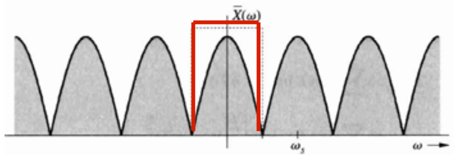
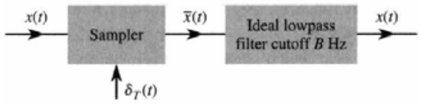
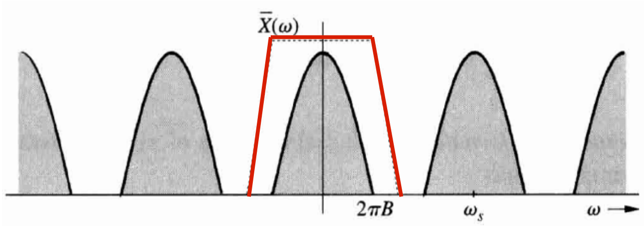
Practical Signal Reconstruction

- Ideal reconstruction system is therefore:





- In practise, we normally sample at higher frequency than Nyquist rate:



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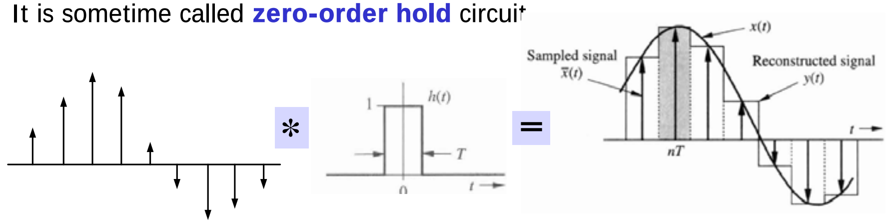
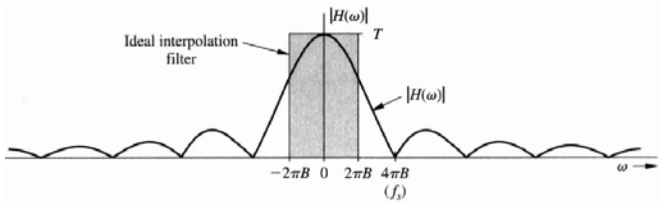
Signal Reconstruction using D/A converter

- D/A converter is a simple interpolator that performs the job of signal reconstruction.
- It is sometime called **zero-order hold** circuit



- The effect of zero-order hold of the D/A converter is a non-ideal lowpass filter.



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